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DYNAMIC ANALOGUES OF SOMIGLIANA'S FORMULA FOR UNSTEADY DYNAMICS OF ELASTIC MEDIA WITH AN ARBITRARY DEGREE OF ANISOTROPY†

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The theory of generalized functions is used to obtain dynamic analogues of Somigliana's formula for the unsteady dynamics of linear elastic homogeneous anisotropic media. Regular integral representations of these analogues are constructed for the solutions of boundary-value problems of the theory of elasticity with homogeneous initial data. Analytic formulae are proposed for the kernels of the integral equations in the case of plane strain for orthotropic media.

Constructions of boundary integral equations for solving static and dynamic boundary-value problems use either Somigliana's formula, which relates the displacements inside a region to the boundary values of the displacements and the loads, or a dynamic analogue of Somigliana's formula. The derivation of these formulae in Nowacki's well-known monograph [1] is based on the Betti reciprocity theorem. We derived a dynamic analogue of Somigliana's formula using the theory of generalized functions [2]. Since the dynamic tensor of fundamental stresses contains non-integrable singularities on the wave fronts, it was necessary to regularize integrals involving such kernels. The problem of regularizing divergent integrals with point singularities was considered, e.g. in [3]. However, regularization of integral equations with singularities on surfaces, as in the theory of boundary-value problems for hyperbolic equations, was first considered in [4]. The case of isotropic media was investigated in [5]. Numerical computations have shown that this regularization technique is convenient for weakly anisotropic media, but presents difficulties for media with strong anisotropy because such media may contain lacunas. We propose here an alternative regular representation, which enables media with an arbitrary degree of anisotropy to be considered.

1. STATEMENT OF THE PROBLEM

Consider a homogeneous elastic anisotropic medium under conditions of plane strain. The equations of motion for such a medium are

$$L_{ij}\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)u_j(\mathbf{x},t)+G_i(\mathbf{x},t)=0,\quad (\mathbf{x},t)\in R_2\times[0,\infty] \tag{1.1}$$

$$L_{ij}\left(\frac{\partial}{\partial \mathbf{x}},\frac{\partial}{\partial t}\right)=C_{imjl}\frac{\partial^2}{\partial x_m\partial x_l}-\rho\delta_i^j\frac{\partial^2}{\partial t^2}\quad (i,j,m,l=1,2)$$

$$C_{ijml}=C_{mlij}=C_{jiml}=C_{ijlm}$$

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where ρ is the density of the medium, G_i are the components of the vector of body forces, and C_{ijml} are the elastic constants of the medium, which constitute a tensor of rank 4 whose indices may be permuted in accordance with the symmetry properties indicated above. Here and below, like-numbered indices indicate summation from 1 to 2.

It is assumed that the domain D is bounded by a Lyapunov contour S [6]. The positive direction along S is chosen so that the medium D should always remain on the left.

We will be concerned with the second boundary-value problem, i.e. we seek a solution of Eqs (1.1) for prescribed boundary values of the non-stationary load vector, which we denote by a function $g_i(\mathbf{x}, t)$ assumed to be bounded for any $(\mathbf{x}, t) \in S \times [0, \infty)$

$$g_i(\mathbf{x},t) = \sigma_{ij}(\mathbf{x},t)n_j(\mathbf{x}) \quad (i,j=1,2) \tag{1.2}$$

where n_i are the components of the unit vector along the outward normal $\mathbf{n}$ to S. The components of the vector of displacements the satisfy initial conditions

$$\partial^m u_i(\mathbf{x},t)/\partial t^m\big|_{t=0} = u_i^m(\mathbf{x}) \quad (m=0,1) \tag{1.3}$$

The functions $u_i^1(\mathbf{x})$, $u_i^0(\mathbf{x})$ $(i=1, 2)$ belong to the class of continuous and continuously differentiable functions in $D: u_i^1(\mathbf{x}) \in C(D)$, $u_i^0(\mathbf{x}) \in C^1(\overline{D})$, and $g_i(\mathbf{x}, t)$ are piecewise-continuous in $\overline{D} \times t$.

2. TENSORS OF THE FUNDAMENTAL SOLUTIONS

We shall consider the system of equations (1.1) in the space of generalized functions $D_2'(R_3)$ defined in the space of compactly supported infinitely differentiable functions $D_2(R_2)$. If $G_i(\mathbf{x}, t) = \delta_i^k \delta(\mathbf{x}, t)$ system (1.1) may be written in a space of integral transforms—a Fourier transformation with respect to the variable $\mathbf{x}$ and a Laplace transformation with respect to the variable t—as follows:

$$L_{ij}(-i\xi,p)\overline{U}_j^k(\xi,p)+\delta_i^k = 0 \tag{2.1}$$

$$L_{ij}(i\xi_1,i\xi_2,p) = C_{imjl}\xi_m\xi_l - \rho p^2\delta_i^j$$

where $L_{ij}(\xi, p)$ are homogeneous polynomials of degree 2 corresponding to the differential operators in (1.1), and $\xi = (\xi_1, \xi_2)$ and P are the parameters of the transforms.

The dynamic Green's tensor $U_j^k(\mathbf{x}, t)$ is a solution of the system of equations (2.1); it is the sum of residues of rational functions [7]

$$U_j^k(\mathbf{x},t) = \frac{1}{\pi t}\operatorname{Im}\sum_{\substack{q=1\\ \operatorname{Im}\zeta_q>0}}^{2} R_{jk\zeta}\left(\zeta_q, 1(x_1\zeta_q + x_2)/t\right) = \frac{1}{\pi t}\sum_{p=1}^{2}\tilde{U}_{jq}^k(\mathbf{x},t) \tag{2.2}$$

$$R_{jk\zeta}(u,\upsilon,w) = \frac{Q_{jk}(u,\upsilon,w)}{Q_{,\zeta}(u,\upsilon,w)}, \quad u=\zeta, \quad \upsilon=1, \quad w=(x_1\zeta+x_2)/t$$

$$Q_{jj}(\xi,p) = -L_{kk}(\xi,p), \quad Q_{jk}(\xi,p) = L_{jk}(\xi,p), \quad j \neq k,$$

$$Q = Q_{11}Q_{22} - Q_{12}^2$$

A comma before an index indicates differentiation with respect to that variable; ζ_p are the roots of the equation

$$Q(\zeta, 1, x_1\zeta + x_2) = 0 \tag{2.3}$$

Green's tensor (2.2) satisfies the conditions

$$U(\mathbf{x}, t) = 0 \quad \text{as} \quad t < 0, \; U(\mathbf{x}, t) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty$$

In (2.2) the function $\tilde{U}_{j1}^k(\mathbf{x}, t)$ corresponds to the quasilongitudinal mode, and $\tilde{U}_{j2}^k(\mathbf{x}, t)$ corresponds to the quasitransverse mode.

The dynamic Green's tensor $U_j^k(\mathbf{x}, t)$ $(j, k = 1, 2)$ of (2.2) generates the tensor of fundamental stresses $S_{ij}^k(\mathbf{x}, t)$ $(i, j, k = 1, 2)$, whose components are determined using Hooke's law

$$S_{ij}^k(\mathbf{x},t) = C_{ijml} \partial U_m^k(\mathbf{x},t) / \partial x_l$$

Besides Green's tensor and the tensor of fundamental stresses, we define further tensors

$$\Gamma_i^k(\mathbf{x}-\mathbf{y},t,\mathbf{n}) = S_{ij}^k(\mathbf{x}-\mathbf{y},t)n_j = C_{ijml} \frac{\partial}{\partial x_m} U_l^k(\mathbf{x}-\mathbf{y},t)n_j$$

$$T_i^k(\mathbf{x},t,\mathbf{n}) = -\Gamma_k^i(\mathbf{x},t,\mathbf{n}) \tag{2.4}$$

The following properties of Green's tensor and the tensor of fundamental stresses will be needed later

$$U_i^k(\mathbf{x}-\mathbf{y},t) = U_i^k(\mathbf{y}-\mathbf{x},t), \; S_{ij}^k(\mathbf{x}-\mathbf{y},t) = -S_{ij}^k(\mathbf{y}-\mathbf{x},t)$$

To construct regular representations of the displacements, it is also convenient to introduce the following convolution tensors

$$V_j^k(\mathbf{x},t) = U_j^m \delta(\mathbf{x},t) * \delta_m^k \delta(\mathbf{x}) H(t) = U_j^k(\mathbf{x},t) * \delta(\mathbf{x}) H(t)$$

$$W_i^k(\mathbf{x},t,\mathbf{n}) = T_i^m(\mathbf{x},t,\mathbf{n}) * \delta_m^k \delta(\mathbf{x}) H(t) = T_i^k(\mathbf{x},t,\mathbf{n}) * \delta(\mathbf{x}) H(t) \tag{2.5}$$

where

$$\delta(\mathbf{x})H(t), \varphi(\mathbf{x},t)) = \int_0^\infty \varphi(0,\tau)d\tau.$$

It follows from (2.5) that

$$\dot{V}_i^k(\mathbf{x},t) = U_i^k(\mathbf{x},t) \tag{2.6}$$

In the space of Fourier–Laplace transforms we have

$$V_j^k(\xi,p) = p^{-1} R_{jk0}(i\xi_1, i\xi_2, p)$$

$$W_j^k(\xi,p) = C_{jsml} i\xi_l n_s p^{-1} R_{mk0}(i\xi_1, i\xi_2 p) \tag{2.7}$$

where

$$R_{jk0}(u,\upsilon,w) = \frac{Q_{jk}(u,\upsilon,w)}{Q(u,\upsilon,w)}$$

By the strong hyperbolicity of (1.1), the characteristic equation

$$Q(i\xi_1, i\xi_2, p) = 0$$

has four pure imaginary roots, in conjugate pairs, which may be written as

$$p_q = i|\xi|c_q, \; \bar{p}_{q+2} = p_q, \; q = 1,2 \tag{2.8}$$

where c_1 and c_2 are the velocities of propagation of quasilongitudinal and quasitransverse modes in the anisotropic medium.

Using the theorem of residues, we determine the inverse Laplace transform with respect to time for the tensors V_j^k, W_j^k

$$L^{-1}\left[V_j^k\right]=\frac{1}{2\pi i}\int_{p-i\infty}^{p+i\infty} p^{-1}R_{jk0}(i\xi_1,i\xi_2,p)\exp(pt)dp=R_{jk0}(i\xi_1,i\xi_2,0)+$$
$$+\sum_{q=1}^{4} ic_q R_{jkc}(i\xi_1,i\xi_2,i|\xi|c_q)\exp(i|\xi|c_q t)$$
$$L^{-1}\left[W_j^k\right]=C_{jsml}i\xi_l n_s L^{-1}\left[V_m^k\right] \tag{2.9}$$

We have used the substitution $p=i|\xi|c$ and the fact that $Q_{,p}=Q_{,c}/\xi$.

Clearly, the first terms in (2.9) are the Fourier transforms of the static tensors of fundamental displacements $U_j^{k(s)}$ and stresses $T_j^{k(s)}$ [8]. The latter is defined by (2.2) with the dynamic Green's tensor U_m^k replaced by the static tensor $U_m^{k(s)}$.

Denote the inverse transforms of the second terms in (2.9) by $V_j^{k(d)}$ and $W_j^{k(d)}$, respectively (the dynamic components). We have

$$V_j^k=U_j^{k(s)}+V_j^{k(d)},\quad W_j^k=T_j^{k(s)}+W_j^{k(d)} \tag{2.10}$$

To determine the inverse transform of V_j^k, it will be convenient to change to a polar system of coordinate system $(\xi,\ \theta):\xi_1=\xi\cos\theta,\ \xi_2=\xi\sin\theta$. Then, taking into consideration that [9, p. 103]

$$\int_0^\infty \alpha^{-1}e^{ib\alpha}d\alpha=-\gamma+i\pi/2-\ln(b+i0)$$

(where γ is Euler's constant), we obtain

$$V_j^k(\mathbf{x},t)=\frac{1}{(2\pi)^2}\int_0^{2\pi}\sum_{q=1}^{4}(ic_q)^{-1}R_{jkc}(\cos\theta,\sin\theta,c_q)\ln\frac{r\cos(\theta-\varphi)}{r\cos(\theta-\varphi)-c_q t}d\theta \tag{2.11}$$
$$r=\sqrt{x_i x_i}\ (x_1=r\cos\varphi,x_2=r\sin\varphi)$$

For the tensor $W_j^k(\mathbf{x},t,\mathbf{n})$ [4]

$$W_j^k(\mathbf{x},t,\mathbf{n})=C_{jsml}\frac{1}{\pi t}\operatorname{Im}\sum_{q=1}^{2}\frac{\delta_1^l\zeta_q+\delta_2^l}{\bar{x}_1\zeta_q+\bar{x}_2}U_{mq}^k(\mathbf{x},t)n_s=$$
$$=C_{jsml}\frac{1}{\pi r}\operatorname{Im}\sum_{q=1}^{2}\frac{\delta_1^l\zeta+\delta_2^l}{\bar{\zeta}_q\cos\varphi+\sin\varphi}R_{mk\zeta}\left(\zeta_q,1,r(\zeta_q\cos\varphi+\sin\varphi)/t\right) \tag{2.12}$$

It follows from the representations (2.11) and (2.12) that

$$V_j^k(\mathbf{x},t)\sim O(\ln r),\ W_j^k(\mathbf{x},t,\mathbf{n})\sim O(r^{-1})\quad \text{as}\quad r\to 0 \tag{2.13}$$

i.e. the asymptotic behaviour of these tensors is determined by that of the static components $U_j^{k(s)}$, $T_j^{k(s)}$ as $r\to 0$.

These tensors are fundamental solutions of the equations of motion (1.1) with the appropriate singular body forces

$$L_{ij}\left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial t}\right)V_j^k(\mathbf{x},t)+\delta_i^k\delta(\mathbf{x})H(t)=0$$
$$L_{ij}\left(\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \frac{\partial}{\partial t}\right)W_j^k(\mathbf{x},t,\mathbf{n})+n_l C_{klmi}\frac{\partial}{\partial x_m}\delta(\mathbf{x})H(t)=0$$

(the indices i,j,k,m,l take values 1, 2).

3. ANALOGUES OF KIRCHHOFF'S AND SOMIGLIANA'S FORMULAE

Let $u(\mathbf{x},t)$ be a solution of Eqs (1.1) in $\{D\times t\}$ satisfying the boundary and initial conditions (1.2) and (1.3). It has been shown [2] that the generalized function $\mathbf{v}(\mathbf{x}, t)=H(t)H_D(\mathbf{x})\mathbf{u}(\mathbf{x}, t)$ satisfies the equations of motion (1.1) with a singular body force and can be written in the form

$$\begin{gathered}\upsilon_i(\mathbf{x},t)=U_i^k(\mathbf{x},t)**F_k(\mathbf{x},t)+\rho U_i^k(\mathbf{x},t)*u_k^1(\mathbf{x})H_D(\mathbf{x})+\rho\dot{U}_i^k(\mathbf{x},t)*u_k^0(\mathbf{x})H_D(\mathbf{x})+\\ +U_i^k(\mathbf{x},t)**g_k(\mathbf{x},t)\delta_S(\mathbf{x})H(t)-C_{kjml}\dot{V}_i(\mathbf{x},t)**\frac{\partial}{\partial x_l}(u_m(\mathbf{x},t)n_j(\mathbf{x})\delta_S(\mathbf{x})H(t)) \qquad (3.1)\\ F_k(\mathbf{x},t)=H(t)H_D(\mathbf{x})G_k(\mathbf{x},t)\end{gathered}$$

A dot over a function indicates differentiation with respect to time, a single asterisk denotes convolution of functions of $\mathbf{x}$, and two asterisks convolution of functions of $\mathbf{x},t$, $g_k(\mathbf{x}, t)\delta_s(\mathbf{x})H(t)$ is a simple layer on the cylinder $\{S*t\}$, $H_D(\mathbf{x})$ is the characteristic function of the set D [8], and υ_i are the components of the vector-valued function $\mathbf{v}(\mathbf{x},t)$.

Because of the presence of strong singularities in the moving wave fronts of the functions $U_{i,l}^k$, $U_{i,t}^k$, it is not possible to transform (3.1) directly to integral notation. This problem has been considered for isotropic elastic media [5]. Regularization of the integrands over the wave fronts enabled an integral form of (3.1) to be obtained for anisotropic media [4], and enabled boundary integral equations to be constructed for solving our boundary-value problem with homogeneous initial data (1.3) [8]

$$u_i^m(\mathbf{x})=0,\ m=0,1 \qquad (3.2)$$

However, this regularization is only feasible for media with weak anisotropy, and presents considerable difficulties in the case of strongly anisotropic media. The regular representation proposed here eliminates this drawback.

Let us consider the most general anisotropy, including even media with strong anisotropy, in which the wave processes typically involve lacunas—moving but undisturbed regions. Such media possess strong waveguide properties in certain directions, and their refraction curves are convex–concave [7].

Using differentiation of convolutions, as well as formulae (2.7) and (3.2), we can write (3.1) in the form

$$\begin{gathered}\upsilon_i(\mathbf{x},t)=U_i^k(\mathbf{x},t)**F_k(\mathbf{x},t)+U_i^k(\mathbf{x},t)**g_k(\mathbf{x},t)\delta_s(\mathbf{x})H(t)-\\ -C_{kjml}\frac{\partial}{\partial x_l}V_i^k(\mathbf{x},t)**\dot{u}_m(\mathbf{x},t)n_j(\mathbf{x})\delta_s(\mathbf{x})H(t)-C_{kjml}\frac{\partial}{\partial x_l}V_i^k(\mathbf{x},t)*u_m^0(\mathbf{x})n_j(\mathbf{x})\delta_s(\mathbf{x}) \qquad (3.3)\end{gathered}$$

By (3.2), the last term in this expression vanishes.

The dynamic Green's tensor (2.2) for an anisotropic medium under plane strain has a weak singularity on the moving wave fronts, i.e. the first two terms on the right of (3.3) have weak singularities on the wave fronts. By (2.5), the same is true of the tensor $W_i^k(\mathbf{x}-\mathbf{y}, t, \mathbf{n}(\mathbf{y}))$.

Thus all the convolutions in (3.3) exist. Using (2.10), we obtain

$$\int_0^t W_i^k(\mathbf{x}-\mathbf{y},\tau,\mathbf{n}(\mathbf{y}))\dot{u}_k(\mathbf{y},t-\tau)d\tau=T_i^{k(c)}(\mathbf{x}-\mathbf{y},\mathbf{n}(\mathbf{y}))u_k(\mathbf{y},t)+\int_0^t W_i^{k(d)}(\mathbf{x}-\mathbf{y},\tau,\mathbf{n}(\mathbf{y}))\dot{u}_k(\mathbf{y},t-\tau)d\tau$$

$$\begin{gathered}H(t)H_D(\mathbf{x})u_i(\mathbf{x},t)=\int_D\int_0^t U_i^k(\mathbf{x}-\mathbf{y},\tau)G_k(\mathbf{y},t-\tau)d\tau\ dD(\mathbf{y})+\int_S\int_0^t U_i^k(\mathbf{x}-\mathbf{y},\tau)g_k(\mathbf{y},t-\tau)d\tau\ dS(\mathbf{y})+\\ +\int_S\int_0^t W_i^k(\mathbf{x}-\mathbf{y},\tau,\mathbf{n}(\mathbf{y}))\dot{u}_k(\mathbf{y},t-\tau)d\tau\ dS(\mathbf{y})\end{gathered}$$

Thus, if (3.2) is true, relations (3.3) for $\mathbf{x}\notin S$ may be written in integral form as follows:

$$H(t)H_D(\mathbf{x})u_i(\mathbf{x},t)=\int_D\int_0^t U_i^k(\mathbf{x}-\mathbf{y},\tau)\,G_k(\mathbf{y},t-\tau)d\tau\ dD(\mathbf{y})+\int_D\int_0^t U_i^k(\mathbf{x}-\mathbf{y},\tau)\,g_k(\mathbf{y},t-\tau)d\tau\ dS(\mathbf{y})+$$

$$+\int_S T_i^{k(c)}(\mathbf{x}-\mathbf{y},\mathbf{n}(\mathbf{y}))\,u_k(\mathbf{y},t)dS(\mathbf{y})-\int_S\int_0^t W_i^{k(d)}(\mathbf{x}-\mathbf{y},\tau,\mathbf{n}(\mathbf{y}))\,\dot{u}_k(\mathbf{y},t-\tau)d\tau\;dS(\mathbf{y}) \quad (3.4)$$

Thus, we have obtained a formula that enables us, given the values of the functions $\mathbf{u}(\mathbf{x},t)$, $\mathbf{u}(\mathbf{x},t)$, $g_i(\mathbf{x}, t)$ on the boundary S, to determine the displacements $\mathbf{u}(\mathbf{x},t)$ in D. Formula (3.4) is a dynamic analogue of Somigliana's formula for anisotropic media.

These formulae involve generalized functions. By the Du Bois–Reymond lemma [10], there is a one-to-one correspondence between locally integrable functions and regular generalized functions. And since the expressions on both sides of the above formulae are regular generalized functions, the equalities are also hold in the usual sense for $\mathbf{x} \notin S$.

In view of the definition of $H_D(\mathbf{x})$, formula (3.4) yields singular integral equations for solving the basic non-stationary boundary-value problems of the theory of elasticity for anisotropic and even strongly anisotropic media.

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